Physica A 525 (2019) 478–490

Contents lists available at ScienceDirect

Physica A

journal homepage: www.elsevier.com/locate/physa

Viral marketing on social networks: An epidemiological perspective

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HIGHLIGHTS

- Novel deterministic and network based approach are proposed to model viral campaign.
- An inert to broadcaster relapse shows longer sustainability of viral message.
- Bistable behavior of message diffusion is observed on ideal and real networks.

ARTICLE INFO

Article history:
Received 30 March 2018
Received in revised form 15 August 2018
Available online 9 March 2019

Keywords:
Viral marketing
Epidemiological model
Bistability
Mean-field analysis
Graph-theoretical treatment
Online social networks

ABSTRACT

Omnipresent online social media nowadays has a constantly growing influence on business, politics, and society. Understanding these newer mechanisms of information diffusion is very important for deciding campaign policies. Due to free interaction among a large number of members, information diffusion on social media has various characteristics similar to an epidemic. In this paper, we propose and analyze a mathematical model to understand the phenomena of digital marketing with an epidemiological approach considering some realistic interactions in a social network. We apply mean-field approach as well as network analysis to investigate the phenomenon for both homogeneous and heterogeneous models, and study the diffusion dynamics as well as equilibrium states for both the cases. We explore the parameter space and design strategies to run an advertisement campaign with substantial efficiency. Moreover, we observe the phenomena of bistability, following which we estimate the necessary conditions to make a campaign more sustainable while ensuring its viral spread.

1. Introduction

In this age of the Internet, the importance of social networks to spread a message, opinion or campaign is undeniable [1–3]. Devising strategies to exploit existing social networks to make a campaign fast spreading as well as sustainable is becoming an area of growing interest among political campaigners and product marketing managers. Based on this ideas, viral marketing (VM) is being adopted as a recent marketing strategy and a way of communication with customers, which can potentially reach a large audience very fast [4–6]. VM is also known as Internet Word-of-mouth marketing, as it encourages people to share information (product specifications, improvements, campaigns etc.) with their friends through

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https://doi.org/10.1016/j.physa.2019.03.008
0378-4371/

email or other social media, and utilizes existing social networks [7]. This prompting is sometimes done by introduction of some benefits (like, credit points, e-cash, extra discounts, cashback, promo codes etc.) to the existing customers, as a reward for sharing information in their peer network.

VM campaigns have several benefits over traditional mass media campaigns, an important one being its ability to reach particular customer groups, as, in many cases, friendship networks arise from common interests [8]. These communications also have more impact and acceptability than third-party advertising among the potential customers, as it comes with an endorsement and recommendation of a friend. Woerdl [9] has also highlighted fast and exponential diffusion among the audience and voluntary transmission by sender as some of the important benefits of viral marketing. The dynamics of VM campaign spread are much similar to that of infectious disease, as they have a contagious quality, being beneficial for the existing consumers and propagating via social interaction. Prominent companies like Amazon, Google and Hotmail have succeeded with virtually no marketing, based solely on consumer-driven communications [10]. In similar fashion, established organizations such as Procter & Gamble, Microsoft, BMW and Samsung have successfully used VM, through which the intact marketing message spreads across the market rapidly, imitating an epidemic [11].

Epidemic models [12] are widely used in different fields of science to study the behavior of different classes of population interacting with each other through a diffusion phenomena [13–16]. These models are commonly based on the idea of contagion through the interaction between people, where the total population is assumed to be compartmentalized. Building differential equations that describe the rate of change in the population class or *compartment*, these models operate on laws outlined by these equations. The underlying assumptions are that the population is homogeneous, people have a constant contact rate and diseases have a unique transmission rate. The SIR model [12], a well-known epidemic model, has three compartments of susceptible, infected, and recovered, and illustrates changes of three compartments using differential equations. Borrowing the concepts from epidemiology, several mathematical models have been proposed by the researchers where spreading is guided by one or more inter-class interactions which could be linear or nonlinear in nature, while a disease, message, habit, addiction, or opinion propagates throughout the population [17–20]. Here we must mention that though compartmental modeling using differential equations is widely used to understand the dynamics of disease spreading, spread of opinion emerges from a complex interplay of information diffusion, individual perception and peer influences. A popular approach to analyze opinion spread follows agent-based modeling (AM). AM also relies on the law of mass action like compartmental modeling [21,22], however it removes the concept of homogeneity that is usually adopted in compartmental disease spreading models. Though, in AM, we can observe the individual status of each participant in a community, being mathematically tractable and computationally inexpensive, compartmental epidemic models have undeniable role in understanding various spreading dynamics. Goffman and Newill [23] carried out the first study on information diffusion using epidemic models by considering the spread of scientific ideas. Based on their work, recently Bettencourt et al. [24,25] further proposed a competency model where two theories simultaneously compete and diffuse in a population. Epidemic models were also tested in diverse problems such as rumor propagation dynamics [26], development of consumer sentiment about economy [27] and prediction of stock buying and selling determinants [28].

While epidemic models, when considered as multivariate differential equations, have a simple treatment with profound understanding, we also have to consider that society has a heterogeneous structure. For this reason, understanding diffusion dynamics is important both from mean-field as well as network perspective. Considering social structures and individuality of the members of a population, a more realistic study can be done using network-based models. In a network model we consider the contact network of individuals inside a population through which diffusion happens and we focus on the effects of network properties in the diffusion process. Barabási and Albert [29] illustrated that diffusion studies using epidemiology models can be successful even on real-world networks, taking certain topological features into account. Independent cascade model (ICM) [30] is a special case of the SIR model which integrates the network structure of the population into the study. A complete digital record of sharing and receiving information from online social networking sites like MySpace, Instagram, Facebook and Twitter have provided a chance to examine the information diffusion in online social media. Utilizing this, Bampo et al. [31] applied the SIR model to various networks to measure the efficiency of email marketing campaigns. In another study with similar motivation, Toole et al. [32] proposed a model to include the effect of geographical and media influences on the adoption of Twitter, by implementing the susceptible–infected–susceptible (SIS) framework. Jalali et al. [33] presented a dynamic model to quantify the core mechanisms of petition diffusion including invitation, interest, awareness, forgetting, sharing and reminding.

All these works establish the importance of including these realistic factors for modeling a real dataset. On the other hand, the more detailed models sometimes become more complex for deeper understanding and application. In this paper, we explore the phenomena of viral marketing from the perspective of effectiveness as well as sustainability, with the help of epidemiological models. While creating a viral ad campaign is a cost effective and fast way to spread the word, in todays' vigorously active social media, there is huge chance of an ad campaign becoming incredibly short-lived. As several recent surveys clearly point out that Internet Word-of-mouth tool is the key to marketing in today's world [34], the importance of quantitative analysis in understanding VM dynamics is becoming undeniable. Majority of studies in this field propose a conceptual framework, completely ignoring the mathematical treatment. A major step forward was the mathematical model for VM dynamics developed by Rodrigues et al. [11], where they attempted to capture the epidemiological aspects and consumer behavior into diffusion dynamics study. However, this treatment could not include some important and evident interactions, which are crucial for dealing with real world scenarios. In this paper, we propose a mathematical model which is rooted in data collected through an extensive questionnaire-based-survey [35], which was aimed towards

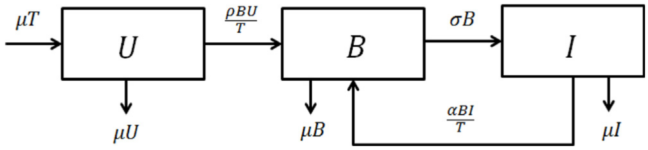

Fig. 1. Block diagram of the proposed model showing all possible transitions from one state to other.

understanding the customer motivation and actual dynamics of VM campaigns. Considering the inputs from real world customers, we build a model resting on the key features observed from the survey outcomes. The precise objective is twofold: first, we consider the case of VM campaign propagation in a population, and propose a simple as well as realistic 3-variable model that shows bistable characteristic. Bistability is an important phenomena having deep implications for the diffusion models where it appears, in the steady state, the number of people in a sub-population can be different depending on the initial state or history of the system. While there could be several contextual as well as creative aspects which can ensure the viral spread of a campaign, we show that some dynamical factors, which are closely entangled with the customer behavior, have major impact on the survival issue. Next, we consider the heterogeneous structure of the society and incorporate the propagation through a network. We compute and simulate for random as well as scale-free networks, and compare the result with real-world social networks. We also demonstrate that strength of the positive feedback in the model is one of the major reasons for bistable behavior of the system which protects the advertisement campaign from a premature death.

2. Proposed model and mean-field analysis

In the mean-field approach, we compartmentalize the total population (T) and associate each individual to one of three mutually-exclusive *subpopulations*: Unaware (U), Broadcaster (B) and Inert (I). This approach has been adopted before for epidemiological modeling [11] as well as network-level treatment [31] of email-based advertisement campaigns. These behavioral transitions between the mutually-exclusive compartments are driven by several guiding factors, which set up the rules for the model. To understand this dynamics in a population and to understand consumer psychology towards VM campaigns, we did an extensive survey-based study in a recent work [35]. Fig. 7(b) and Table 1 of [35] summarizes the findings of that survey in the form of a theoretical framework. There we observed that there are several inter-dependent and independent factors that play major roles in customer's approach towards a VM campaign. As discussed in Sec. 5 of [35], reasons like amount of rewards, recent trends and association with a brand name etc., motivate people to participate actively in a viral campaigning. However, even after knowing about an offer, sometimes people do not participate in it for several reasons like security, forgetting, adverse past experience etc. The survey also shows that proof of genuineness of the campaign and friendly reminders can engage the noncontributing subpopulations in the diffusion of the viral offer. Taking these aspects into consideration, and also assuming homogeneous mixing i.e., overlooking the spatial structure within the population, we describe the system as follows.

2.1. Model formulation

The scheme of the model is illustrated in Fig. 1. The unaware class, denoted by U, is yet to receive the message; these are susceptible people or the target market, who may receive an advertising message containing marketing offers. The broadcaster class B consists of individuals who came to know about the message and have the potential to forward the message further in the population. If a member in this class decides to participate in the campaigning, (s)he spreads and transmits the message in the entire population by recommending it through their social contacts. We assume that ρ is the rate at which a broadcaster comes in contact with a member from unaware class, and share the viral message to create new potential broadcasters. Finally, there is the inert class I, who used to be in broadcaster class B, but is not sharing the message at present. There are two major ways into the inert class: first, people who used to broadcast the viral message can lose interest and come to inert class due to several factors like, getting annoyed (due to low profit-to-effort ratio), bored or suddenly doubtful (about security). On the other hand, some people, who were in the B class and had the option to share the viral message, may directly come in the inert class without sharing the message even once. The factors that can influence this event are forgetting, diversion, safety concerns, insecurity about hidden clauses etc. Effects of these factors are combined in the parameter σ of the model.

Interestingly, in [35] we found that there are always chances that the inert may regain their interest depending on the reasons that brought them to the inert class I in the first place. With 331 participants sharing their opinions through both polar as well as qualitative answers in that survey, we specifically focus on the two kinds of people who transit from B to I class to get a clear idea of what makes them gain their interest back in the campaign. The first kind who had left the B class by getting bored or annoyed, more than 92% of them agreed that an authentic information or a genuine news related to a considerable gain from the same campaign might make them motivated to return to active participation [35]. In contrast, the other subgroup, where people might have switched to the inert class shortly after getting the viral message, had an initial interest about the offer. The driving causes for them are entirely different, like, forgetting or attention diversion. These people, who often feel that they *missed* the offer (almost 76%), continue to be submissively interested about the advertisement campaign. A timely reminder by someone or strategically designed retargeting emails from the company can influence them straightaway to gain their active state back, and they start contributing in the propagation of the campaign again. It has been also seen that coming across frequent discussions about a product or campaign (commonly referred as *buzz*) can tempt the inerts to become a broadcaster again. All these possible transitions from I to B have been included in our model by allowing a feedback from inert class to broadcaster with a relapse rate α.

In practical scenarios, people enter and leave the population. To include this factor, we have introduced birth and death in our model. Both birth and death rates are kept equal to μ, so that a fixed population size can be maintained [11,16,21,22,27,31]. For a particular VM dynamics, birth and death can be viewed as events when people join or leave a particular social platform where the campaign is going on. Considering u, b and i to be the fraction of unaware, broadcaster and inert classes normalized by the total population T, the VM dynamics in the population with the mentioned interactions is governed by the following differential equations:

$$\begin{aligned} u' &= \mu - \rho bu - \mu u \\ b' &= \rho bu - \sigma b - \mu b + \alpha bi \\ i' &= \sigma b - \alpha bi - \mu i \end{aligned} \tag{1}$$

Here and at all subsequent places, u' and all such terms denote the rate of change with time.

2.2. Equilibrium analysis

At equilibrium there is no time evolution of the system model defined in Eq. (1) and the rate of change of u, b and i become zero. The system of equations always has a VM-free equilibrium E_0, at which the whole population is unaware. Also, the system exhibits an Endemic equilibrium $E^\star$ with a finite percentage becoming broadcasters. By setting the u', b' and i' of Eq. (1) to zero, all the components of $E^\star$ can be evaluated.

2.2.1. Stability

It is necessary to find out the stability of the equilibria to interpret the dynamics of the system. Linear stability analysis is a straight forward way for classifying equilibrium points under small perturbation. Let us consider a set of ordinary differential equations, $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ with an equilibrium point $\mathbf{x}^\star$. We can linearize the equation by Taylor series expansion around the equilibrium as,

$$\mathbf{f}(\mathbf{x}) = \mathbf{f}(\mathbf{x}^\star) + \left.\frac{\partial \mathbf{f}}{\partial \mathbf{x}}\right|_{\mathbf{x}^\star} (\mathbf{x} - \mathbf{x}^\star) \tag{2}$$

On the other hand, we can consider a small perturbation $\delta\mathbf{x}$ from the steady state by letting, $\mathbf{x} = \mathbf{x}^\star + \delta\mathbf{x}$. In this condition, the question of stability translates into the eventual decay (or growth) of $\delta\mathbf{x}$, so that $\mathbf{x}$ comes back to (or moves away from) the steady state $\mathbf{x}^\star$, by deciding it to be a stable (or unstable) solution. So, to study the behavior of $\delta\mathbf{x}$ with time, we take a time derivative, to find that,

$$\delta\dot{\mathbf{x}} = \dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}), \tag{3}$$

as $\mathbf{x}^\star$ is a constant. Drawing the equivalence between Eqs. (2) and (3) as both express a form of $\mathbf{f}(\mathbf{x})$, we write,

$$\delta\dot{\mathbf{x}} = J^\star \delta\mathbf{x}, \tag{4}$$

where $J^\star$ is the Jacobian evaluated at the equilibrium. For that equilibrium $\mathbf{x}^\star$ to be *stable*, *all* the eigenvalues of $J^\star$ have *negative* real part. For a system of N ordinary differential equations, where N variables are coupled with each other, the components of state vector $\mathbf{x}$ are $[x_1, x_2, x_3 ..., x_N]$ and the components of rate vector $\mathbf{f}$ are $[f_1, f_2, f_3,..., f_N]$; in this case, the Jacobian is

$$J^\star = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_N} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_N} \\ \cdots & \cdots & \cdots & \\ \frac{\partial f_N}{\partial x_1} & \frac{\partial f_N}{\partial x_2} & \cdots & \frac{\partial f_N}{\partial x_N} \end{bmatrix} \tag{5}$$

evaluated at $[x_1^\star, x_2^\star, x_3^\star, \ldots, x_N^\star]$. For analyzing the stability of the fixed points of our model we have,

$$\begin{aligned} f_1 &= \mu - \rho b u - \mu u \\ f_2 &= \rho b u - \sigma b - \mu b + \alpha b i \\ f_3 &= \sigma b - \alpha b i - \mu i \end{aligned} \tag{6}$$

We obtain $J^\star$ using Eq. (5) for all the steady states to analyze the eigenvalues and determine the stability of the steady states.

2.2.2. Bifurcation

While solving for $E^\star$, the first equation of system model, defined in Eq. (1), gives

$$u^\star = \frac{\mu}{\rho b^\star + \mu} \tag{7}$$

Relevant substitutions from Eq. (7) and replacing $i^\star$ by $(1 - b^\star - u^\star)$, simple algebra results into $p(b^\star)^2 + qb^\star + r = 0$, where

$$p = \alpha\rho;\ \ q = (\sigma\rho + \mu\rho + \alpha\mu - \alpha\rho);\ \ r = \mu(\sigma + \mu - \rho) \tag{8}$$

Examining the coefficients, we conclude that p is always positive; q is positive for small values of α, and r is positive or negative depending on whether $\frac{\rho}{\sigma+\mu} = \mathcal{R}$ is smaller or greater than 1. Two utterly different steady state scenarios can arise:

Case 1: For negative r (i.e., $\mathcal{R} > 1$), the quadratic equation has a unique positive solution $b_+^\star$, as another solution $b_-^\star$ is always negative and so, unphysical, and there exists a unique endemic equilibrium $E^\star$ whenever $\mathcal{R} > 1$.

Case 2: On the other hand, for positive r (i.e., $\mathcal{R} < 1$), the number of physical roots of the equation depends on the sign of q, and therefore, the nonlinear relapse parameter α. Depending on this fact if α is high (or low), multiple (or no) endemic equilibria may exist.

To understand the phenomenon, we observe the steady states of the model for two different α values fixing $\mu = 0.05$ and $\sigma = 0.2$ in Fig. 2. A forward transcritical bifurcation is observed in Fig. 2(a), indicating the existence of only the message-free solution before $\mathcal{R} = 1$. On the other hand, a backward bifurcation occurs for high α values as shown in Fig. 2(b). In this case, for the parameter regime where $\mathcal{R} \in [\mathcal{R}_c, 1)$, there exists a choice for the system between two distinctly different responses. This regime is known as region of bistability where both the endemic and the message-free solutions can be achieved by the system depending upon the initial conditions. This history dependence is commonly known as *hysteresis*, drawing an analogy from the ferromagnetic systems. This phenomena of bistability gives the system a sustainability, so that, once a transition occurs from the message-free state to the endemic state, the nonlinearity of the dynamics inherently opposes any switch-back driven by the immediate fluctuations of the parameters; the whole system works as a very robust switch. Thus, it can be concluded that *high value of nonlinear relapse rate α makes it difficult to eradicate the message from the system;* the message-epidemic will be present for a broad parameter regime, even when $\mathcal{R} < 1$. The condition for existence of this bistable region will be discussed in Section 2.4.

2.3. Reproduction number

It is evident that the quantity $\mathcal{R}$, is dictating the basic behavior (message-free or endemic) of the system. The concept could be understood more clearly just from the structure of the model. Here, $\mathcal{R}$ is analogous to the basic reproduction number defined for SIR model, which is average number of broadcasters a single broadcaster can create in its lifetime *without* considering its interaction with inert class (which makes α irrelevant for this estimation). The rate at which a broadcaster will interact with unaware is ρ. A broadcasters' lifetime can have two components: μ, for natural death rate and σ, for death-like conversion to inert class. So, the average lifetime will be $\frac{1}{\mu+\sigma}$ and the average number of other broadcasters created in this lifetime will be $\frac{\rho}{\mu+\sigma}$.

2.4. Conditions for bistability

To ensure bistability, the necessary conditions are $q < 0$ and $q^2 - 4pr > 0$ where, p, q, r are given by Eq. (8). We can figure out the limiting condition for bistability from these relations. The nonlinear relapse rate, α causes the drastic change in the behavior of the system, though it does not appear in the expression of $\mathcal{R}$. By equating $q = 0$, we can figure out the minimum threshold for α as:

$$\alpha_{th} = \frac{\rho(\sigma + \mu)}{\rho - \mu} \tag{9}$$

For a given set of parameters, *iff* $\alpha > \alpha_{th}$, then bistable solutions can be expected. Once we satisfy the condition for α, it should be noted that both the endemic states can exist (i.e., have real roots) only if $q^2 - 4pr > 0$. The region of bistability extends for a range of $\mathcal{R}$ values, from $\mathcal{R}_c$ to 1, as mentioned before. For $\mathcal{R} < \mathcal{R}_c$ neither of the endemic solutions are

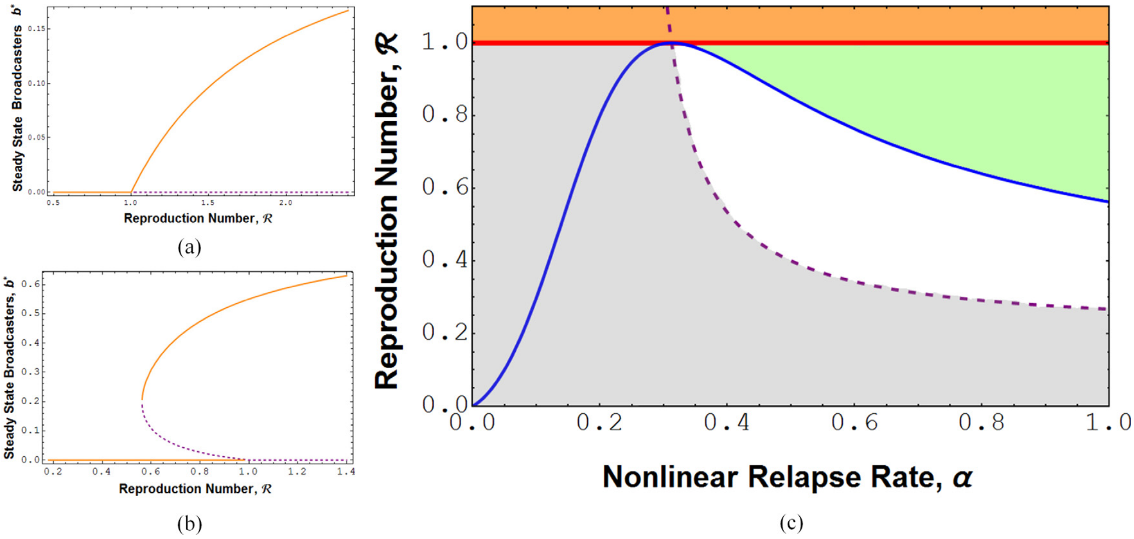

Fig. 2. Variation in steady state fraction of b with reproduction number $\mathcal{R}$ for (a) $\alpha = 0.1$, when only a single epidemic state persists beyond $\mathcal{R} = 1$ and for (b) $\alpha = 1$, when bistability can be observed in range $\mathcal{R}_c$ to 1. In these figure, orange (and continuous) lines indicate stable solutions and purple (and dashed) lines indicate unstable solutions. For these parameter values, we calculated $\mathcal{R}_c = 0.562$ from Eq. (10). (c) Phase diagram of the model in $\alpha - \mathcal{R}$ space for $\sigma = 0.2$ and $\mu = 0.05$. The blue line indicates $\mathcal{R}_c$, purple dashed line indicates α_{th} and red line indicates $\mathcal{R} = 1$. The region filled with orange color always exhibits monostable endemic state as $\mathcal{R} > 1$, the gray region exhibits monostable VM free state as $\alpha < \alpha_{th}$. For both the white region and green region, $\alpha \geq \alpha_{th}$. For the white region, $\mathcal{R} < \mathcal{R}_c$ and the region contains monostable VM free state. However for the green region, $\alpha > \alpha_{th}$, $\mathcal{R} > \mathcal{R}_c$ and $\mathcal{R} < 1$. Thus, this area exhibits bistability, where either VM free state or the endemic state is chosen by the system depending upon the initial state.

feasible and the only steady state is message-free. $\mathcal{R}_c$, or the critical threshold for bistability can be evaluated by equating, $q^2 - 4pr$ to zero. With algebraic manipulations, we can show that

$$\mathcal{R}_c = \frac{1}{(\sigma + \mu)} \frac{\alpha\mu}{(\alpha + \mu + \sigma - 2\sqrt{\sigma\alpha})} \tag{10}$$

Eqs. (9) and (10) provide us with two limits for ensuring the bistable dynamics of the system.

From our previous discussions, it is evident that depending upon the two key parameters of the model, $\mathcal{R}$ and α, *only* endemic, *only* VM free or *both* solutions can be obtained. To illustrate this idea, we explored the phase diagram of the system in $\alpha - \mathcal{R}$ space in Fig. 2(c). The only region where bistable dynamics can be observed is shaded in green, bounded by the $\mathcal{R} = 1$, Eqs. (9) and (10). Fig. 2(a)(and (b)) can be obtained by tracking the system behavior while moving across the phase space through a vertical line $\alpha = 0.1$ (and $\alpha = 1$). We note that region of bistability increases gradually as the value of α increases. This shows that high values of the relapse rate ensure the survival of the campaign in steady state. From the phase diagram, it can also be noted that without the relapse (i.e., $\alpha = 0$), no bistability is possible.

3. Graph-theoretical analysis

In contrast to mean-field approach, diffusion in networks will be dependent on the degree distribution of the network. We denote with u_k, b_k and i_k the fraction of unaware, broadcaster and inert nodes with degree k. Equations now modifies to:

$$\begin{aligned} u'_k &= \mu - \beta k u_k b - \mu u_k \\ b'_k &= \beta k u_k b + \gamma k i_k b - (\sigma + \mu) b_k \\ i'_k &= \sigma b_k - \gamma k i_k b - \mu i_k \end{aligned} \tag{11}$$

We are considering β to be the rate at which a broadcaster spreads the information to an unaware neighbor. Similarly, γ is the relapse rate influenced by the neighbors. In these equations we have assumed that the fraction of broadcasters around a node of degree k is independent of k. But, in a real network this is not the case. Θ_{k_b} is the density function which gives probability of broadcasters around a node of degree k. In uncorrelated network Θ_{k_b} is independent of k and is given by

$$\Theta_b = \sum_k \frac{k p_k b_k}{\langle k \rangle} \tag{12}$$

Substituting Θ_b in Eq. (11), we get

$$\begin{aligned} u_k' &= \mu - \beta k u_k \Theta_b - \mu u_k \\ b_k' &= \beta k u_k \Theta_b + \gamma k i_k \Theta_b - (\sigma + \mu) b_k \\ i_k' &= \sigma b_k - \gamma k i_k \Theta_b - \mu i_k \end{aligned} \tag{13}$$

Multiplying all these three equations by $\frac{kp_k}{\langle k \rangle}$ and then performing summation over k, we get

$$\begin{aligned} \Theta_u' &= \sum_k \frac{kp_k}{\langle k \rangle}\mu - \beta \sum_k \frac{k^2 p_k}{\langle k \rangle} u_k \Theta_b - \mu \sum_k \frac{kp_k}{\langle k \rangle} u_k \\ \Theta_b' &= \beta \sum_k \frac{k^2 p_k}{\langle k \rangle} u_k \Theta_b + \gamma \sum_k \frac{k^2 p_k}{\langle k \rangle} i_k \Theta_b - (\sigma + \mu) \sum_k \frac{kp_k}{\langle k \rangle} b_k \\ \Theta_i' &= \sigma \sum_k \frac{kp_k}{\langle k \rangle} b_k - \gamma \sum_k \frac{k^2 p_k}{\langle k \rangle} i_k \Theta_b - \mu \sum_k \frac{kp_k}{\langle k \rangle} i_k \end{aligned} \tag{14}$$

3.1. Propagation at initial state

In initial phase of message spreading [36], b and i can be approximated by zero and u by 1. Using these values in nonlinear terms so that they can be simplified to linear equation, we get

$$\begin{aligned} \Theta_u' &= \mu - \beta \frac{\langle k^2 \rangle}{\langle k \rangle} \Theta_b - \mu \Theta_u \\ \Theta_b' &= \beta \frac{\langle k^2 \rangle}{\langle k \rangle} \Theta_b - (\sigma + \mu) \Theta_b \\ \Theta_i' &= \sigma \Theta_b - \mu \Theta_i \end{aligned} \tag{15}$$

Integrating second equation of the above system and using b_0 as initial value of Θ_b, we get $\Theta_b = b_0 e^{\frac{t}{\tau_b}}$ where

$$\tau_b = \frac{\langle k \rangle}{\beta \langle k^2 \rangle - (\sigma + \mu) \langle k \rangle} \tag{16}$$

Putting value of Θ_b in third equation of the system and using i_0 as initial value of Θ_i, we get $\Theta_i = C_1 e^{\frac{t}{\tau_b}} + C_2 e^{-\frac{t}{\tau_i}}$, where

$$C_1 = \sigma \tau_b b_0; \;\; C_2 = \sigma \tau_b b_0 - i_0; \;\; \tau_i = \frac{1}{\mu} \tag{17}$$

Similarly first equation of the system with u_0 as initial value of Θ_u gives $\Theta_u = \mu t + C_3 e^{\frac{t}{\tau_b}} + C_4 e^{\frac{-t}{\tau_u}}$, where

$$C_3 = -\beta \frac{\langle k^2 \rangle}{\langle k \rangle} \tau_b b_0; \;\; C_4 = u_0 + \beta \frac{\langle k^2 \rangle}{\langle k \rangle} \tau_b b_0; \;\; \tau_u = \frac{1}{\mu} \tag{18}$$

For epidemic to spread, τ_b must be positive. This condition gives a relation between epidemic and network parameters to ensure epidemic i.e.,

$$\frac{\beta}{\sigma + \mu} > \frac{\langle k \rangle}{\langle k^2 \rangle} \tag{19}$$

Expression on left hand side of the equation is similar to that of the Reproduction number of the mean-field treatment. For Erdös–Rényi Random Network [37], $\langle k^2 \rangle = \langle k \rangle (\langle k \rangle + 1)$ and hence threshold is $\frac{1}{\langle k \rangle + 1}$ which decreases with increase in average degree. For scale-free network with degree exponent in the range (2,3] $\langle k^2 \rangle$ diverges which leads to absence of epidemic threshold [38,39].

3.2. Steady state network analysis

In large time limit, system will reach steady state. Rate of change of fractions u, b and i will be zero. In case of degree based compartment scheme, u_k, b_k and i_k will not change. Equating all three system evolution equations (Eq. (13)) to zero, we have

$$b_k = \frac{\beta k \Theta_b (\mu + \gamma k \Theta_b)}{(\mu + \beta k \Theta_b)(\gamma k \Theta_b + \sigma + \mu)} \tag{20}$$

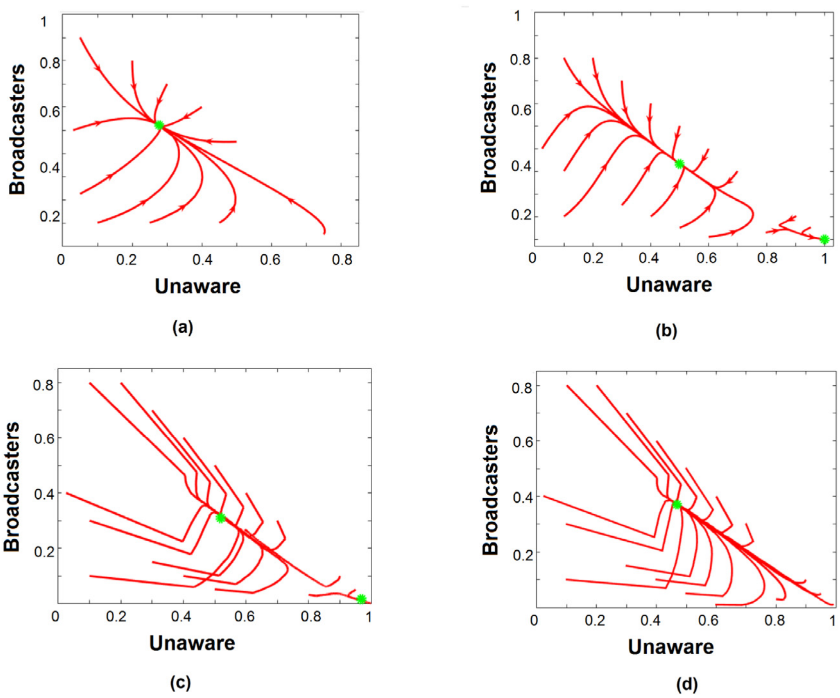

Fig. 3. Numerical simulation of convergence to the steady state for different initial conditions, for (a) deterministic mean-field system with single endemic steady state for $\mu = 0.05$, $\sigma = 0.1$, $\rho = 0.25$ and $\alpha = 0.4$; (b) Deterministic mean-field system with bistable dynamics for $\mu = 0.05$, $\sigma = 0.15$, $\rho = 0.15$ and $\alpha = 0.75$; Temporal variation of u and b with different initial conditions for equivalent parameter regime as (b) in (c) random network and (d) scale-free network. To ensure the equivalence with the mean-field analysis, the infection rate and the relapse rate for network dynamics are taken as $\rho/\langle k\rangle$ and $\alpha/\langle k\rangle$ respectively.

Multiplying b_k by $\frac{kp_k}{\langle k\rangle}$ and performing summation over k we get

$$\Theta_b = \frac{1}{\langle k\rangle}\sum_k \frac{p_k k^2 \beta \Theta_b(\mu + \gamma k\Theta_b)}{(\mu + \beta k\Theta_b)(\sigma + \mu + \gamma k\Theta_b)} \tag{21}$$

This is a self consistency equation where $\Theta_b = f(\Theta_b)$. At $\Theta_b = 0$; $f(\Theta_b)$ is also zero. Hence $\Theta_b = 0$ is a solution of the equation. Value of the function at $\Theta_b = 1$ is

$$f(1) = \frac{1}{\langle k\rangle}\sum_k \frac{p_k k}{(1 + \frac{\mu}{\beta k})(1 + \frac{\sigma}{\mu + \gamma k})} \tag{22}$$

It is clear from the above expression that $f(1) < 1$. To have another solution in the interval 0 to 1, slope of the function at $\Theta_b = 0$ must be greater than 1.

$$\frac{\mathrm{d}f(\Theta_b)}{\mathrm{d}\Theta_b}\Big|_{(\Theta_b=0)} = \frac{1}{\langle k\rangle}\sum_k \frac{p_k k^2 \beta}{(\sigma + \mu)} = \frac{\beta}{(\sigma + \mu)}\frac{\langle k^2\rangle}{\langle k\rangle} \geq 1$$

which is the same condition we had from linear approximation in initial phase of the epidemic.

4. Numerical results

Simulations for both the approaches, mean-field analysis and network analysis have been carried out on MATLAB. To understand the effect of population heterogeneity and network topology, random, scale-free and real social networks have been considered in our simulations.

4.1. Simulation of the deterministic model

As discussed in Section 2.4, system may lead to any one of three possible steady state situations: when only message-free state exists; when only an endemic state exists; when an endemic as well as a message-free state can exist depending on initial population of different classes. We have shown both the cases where at least one of the steady state is endemic, in Fig. 3(a)–(b) with their respective parameter values.

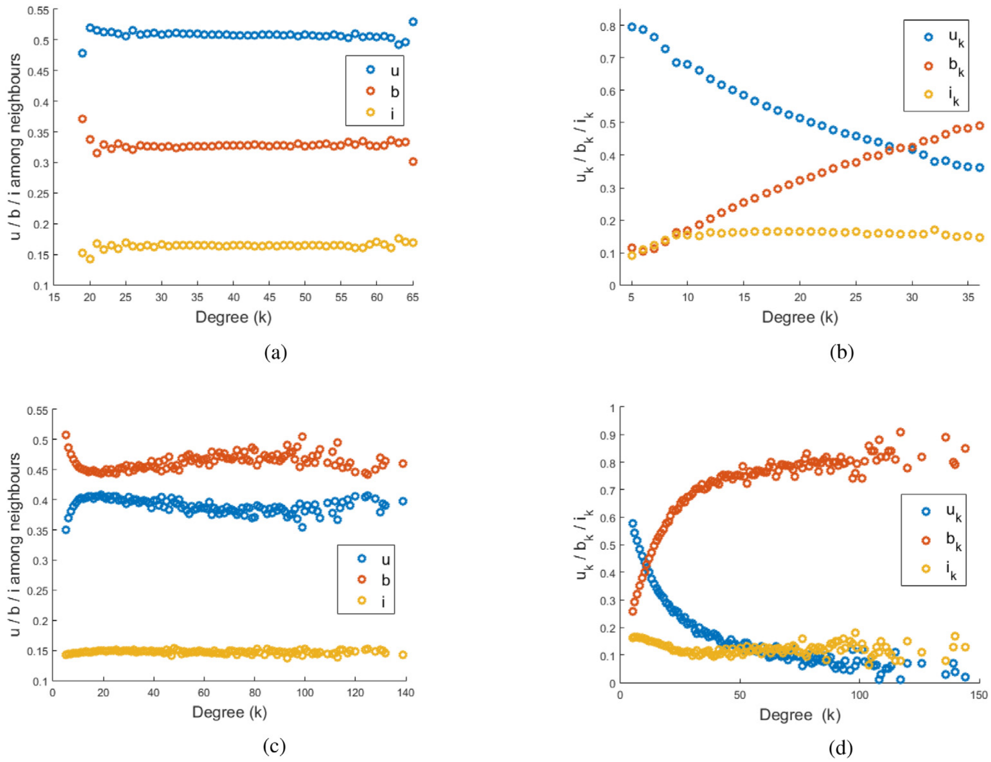

Fig. 4. (a) Fraction of u, b and i in the neighborhood of a node with degree k in random network; (b) u_k, b_k and i_k with respect to k at steady-state in random network; (c) Fraction of u, b and i in the neighborhood of a node with degree k in scale-free network; (d) u_k, b_k and i_k with respect to k at steady-state in scale-free network.

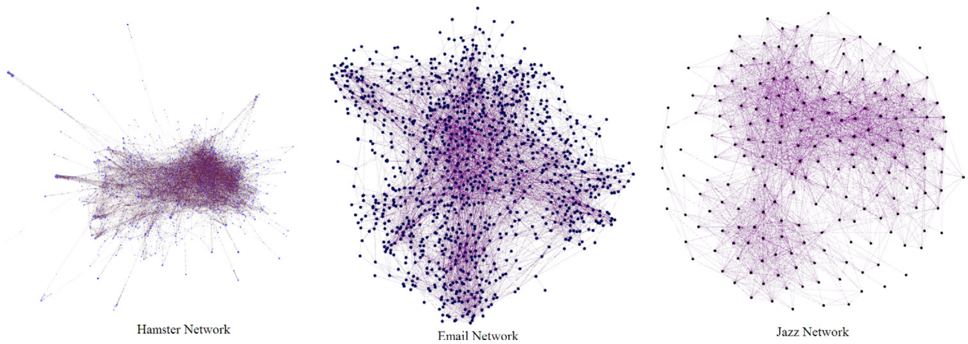

Fig. 5. Physical topologies of real networks used in our work.

4.2. Simulation over model networks

Same set of parameters have been used to compare the results of deterministic mean-field model with network model. We have carried out our simulation over random network and scale-free network having 1024 nodes and average degree 10. Random network has been created by Erdös–Rényi model and follows binomial degree distribution which converges to Poisson distribution for very large number of nodes (infinite network). Scale-free network has been generated by Barabási–Albert preferential attachment model [29] and follows power law degree distribution having power exponent 3.

In case of random network, we obtained similar results as predicted by deterministic model. Using the values of bistable steady state configuration of Fig. 3(b), we similarly observed two stable states in Fig. 3(c), one endemic and one message-free. Steady state value obtained by the simulation is also in agreement with the mean-field approach.

Table 1
Important characteristics of different network.

Network characteristics	Hamster network	Email network	Jazz network
Number of nodes	2426	1133	198
Number of edges	16 631	5451	2742
Average degree	13.71	9.624	27.7
Maximum degree	273	71	100
Power law exponent	2.46	6.77	5.27

In case of scale-free network, endemic steady state values are not exactly same. In considered set of parameters, there is maximum 4% error in steady state fraction of different classes. This error is due to non-homogeneity present in scale-free network in form of hubs. Second point of difference is that message-free state never appears in scale-free network. The fact is in alignment with our analytic result regarding absence of epidemic threshold in scale-free network. It can be observed in Fig. 3(d) where temporal variation of u and b leads to endemic steady state in every set of initial conditions, in contrast with random network scenario of Fig. 3(c), where we can see few flows terminating at message-free steady state.

To understand the dynamics of nodes of different degree, we have plotted steady state value of u, b and i in their neighborhood. As shown in Fig. 4(a), for random network, these fractions are independent of degree of nodes. Hence, the number of broadcasters around a higher degree node is large as compared to a lower degree node which makes them more prone to receive the message. It is evident from Fig. 4(b) where fraction of broadcasters is shown to be monotonically increasing with degree. Even in scale-free network, b_k increases with degree k as shown in Fig. 4(d). But, in Fig. 4(c) fraction of broadcasters around any node is more than fraction of unawares which is entirely opposite to the random network case shown in Fig. 4(a). This outcome is result of these two features of scale-free network: (i) chances of higher degree nodes (hubs) getting infected is very high as shown in Fig. 4(c) and (ii) same hubs are present in neighborhood of multiple nodes while counting broadcasters around a node. This redundancy leads to increment in local fraction of broadcasters in neighborhood of a node.

4.3. Simulation over real networks

Though Figs. 3 and 4 indicate that the information diffusion in the proposed model follow the dynamics as discussed in Sections 2 and 3, it is important to analyze the model over real world networks as most of these networks do not follow the typical characteristics of any particular model network. Thus, we have studied the proposed viral marketing model over some real networks collected from KONECT database [40], to understand its behavior in real social interactions scenarios.

The networks that we have used for testing our VM models are referred as Hamster network, Email network and Jazz network in the rest of the paper. The first real network that we considered is the friendship network of website www.hamsterster.com that has 2426 users (nodes) with 16,631 friendships edges. The second one, the Arena email network has been collected from University Rovira i Virgili of Spain. The network has 1133 users (nodes) with 5451 connections (edges). The third one, referred as the Jazz network, is the collaboration network between jazz musician that can be visualized as a network of people with common interest or skill. The jazz network has 198 musicians (nodes) with 2742 collaborations (edges). In Fig. 5 we show the topologies of the real networks used in this work. The network parameters for the real networks that are considered in this paper are summarized in Table 1.

We study the flow of a viral message in all three real networks mentioned above. As email network has almost same number of nodes as the model networks considered in our simulations, we show the time evolution of email network for different parameters and initializations in Fig. 6 as the message diffuses. In Fig. 6(a), we set the parameters equivalent to Fig. 2(a) along with $\mathcal{R} = 0.64$, which gives only VM-free state in mean field analysis. For all different initializations, we get a complete broadcaster-free state in the email network as well. To analyze bistability, we set the parameters equivalent to Fig. 2(b) with $\mathcal{R} = 0.64$, which belongs to a bistable region in deterministic case. To locate the lower branch, we generated five different realizations of simulations for 10^4 time units, and results were obtained where 2% of the nodes were broadcasters initially. If the infected fraction went to zero in any of the five runs, the message-free state was considered stable [41]. Simulation results show that the email network exhibits VM-free state when the number of broadcasters is low initially. To locate the upper branch, the system was run to steady state where initially 70% of the total population were acting as broadcaster and none were in inert state in each network. For each run, we studied the system for 10^4 time units and then averaged over 200 samples. Like the deterministic model, with high number of broadcasters initially, the email network also shows endemic state. We show the steady states of the email network for both the initializations in Fig. 6(b) and (c) respectively; different final states for different initial conditions demonstrates the existence of hysteresis. It is also noted that for all different networks, systems' propensity for the endemic state starts to dominate as $\mathcal{R}$ goes beyond $\mathcal{R}_c$, for a specific parameter set. As $\mathcal{R}_c$ is always less than 1, even in real networks we acquire a state of endemic for $\mathcal{R} < 1$, where the message is being spread throughout the population. The final steady state conditions for the real networks are compared in Table 2 for $\mathcal{R} = 0.64$ with 70% broadcasters initially and parameters equivalent to Fig. 2(b). It shows that 30%–35% of the population belong to the broadcaster class, ensuring the survival of the advertisement campaign in steady state, even when $\mathcal{R} < 1$.

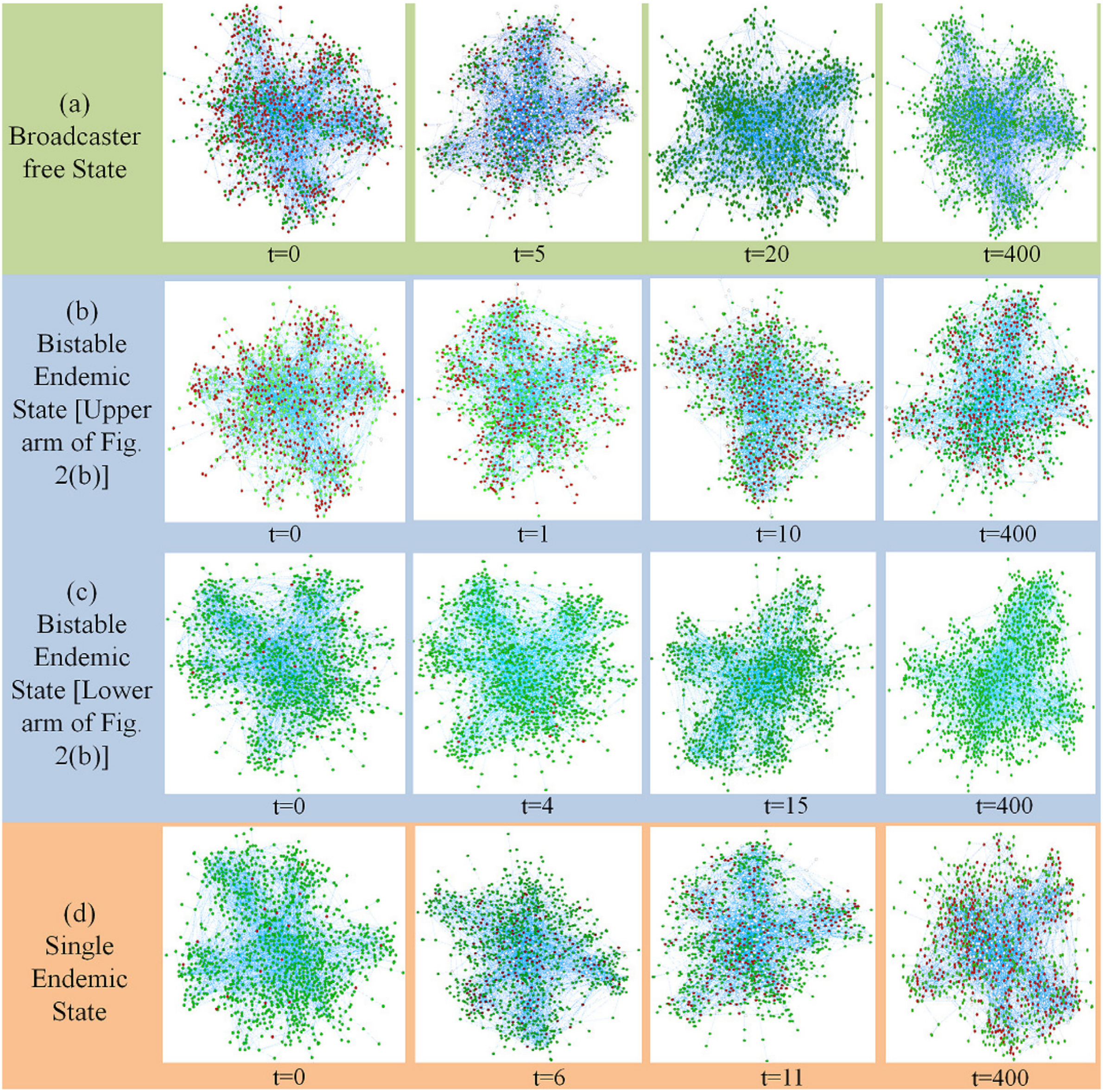

Fig. 6. Time evolution of email network for $\mathcal{R} = 0.64$: (a) when the network parameters are equivalent to Fig. 2(a); steady state is completely free of broadcasters. (b) When the network parameters are equivalent to Fig. 2(b), which satisfies $\mathcal{R}_c < \mathcal{R} < 1$ with 70% broadcasters initially; steady state is endemic, having 30% broadcasters. (c) When the network parameters are equivalent to Fig. 2(b) but with 2% broadcasters initially; steady state is completely free of broadcasters. (d) When the network parameters are equivalent to Fig. 2(b) but for $\mathcal{R} = 1.4$, that satisfies $\mathcal{R} > 1$ with 2% broadcasters initially; steady state is endemic. green, red and white colors represent unaware, broadcaster and inert nodes respectively. Please refer to the online version of the paper at maximum zoom to fully appreciate the results.

Table 2
Comparison of viral message diffusion in different networks for $\mathcal{R} = 0.64$.

Steady state fraction	Random network	Scale-free network	Hamster network	Email network	Jazz network
u^*	0.54	0.49	0.59	0.53	0.50
b^*	0.32	0.37	0.31	0.35	0.37
i^*	0.14	0.14	0.1	0.12	0.13

5. Conclusion

Marketing is always considered as one of the key components, not an auxiliary arrangement, for a successful business [42]. Surely, a viral marketing campaign works as a less expensive and unexpected way to reach the customers; but nowadays, when almost 85 million videos and photos get uploaded every day in a popular social networking website like Instagram [43], the main challenge is making that advertisement execute long lasting iterations in the population, so that it can reach a bigger audience. If we consider just the case of Instagram, when almost 500,000 advertisers are using this popular website for campaigning [44], most of the uploads get tossed like a needle in a haystack. The model we propose in this paper establishes a principle of sustainability for online advertisement campaigns by developing a model relying on rigorous consumer psychology survey data [35]. Extensive analysis with mean-field equations as well as networks simulations shows that the region of bistability grows as the value of α, the nonlinear relapse rate increases. Bistability gives the system a chance to retain its viral state for adverse parametric conditions as well. While we have observed that the steady states of diffusion in networks are closely related to mean-field system dynamics, we also appreciated the importance of network structures in this issue. Not only on model systems, but in real social networks, with consideration of all heterogeneity that exists in a population, it has been shown that sustainability of a viral campaign is actually dependent on drawing the attention of those who are not participating in spite of being aware of the campaign. If a certain percentage of this inert population start broadcasting in favor of the campaign, it retains its endemic state in the entire population.

It is to be noted that for a more realistic modeling of the dynamics, the birth and death rates could be considered different, i.e., we can assume that people enter and leave the population in different rates. Considering that in a social media platform, new people migrate in at a much faster rate than the rate at which people leave the platform, it might be assumed that the birth rate of unaware people is μ while the death rate for them is μ_1 ($< \mu$). Death rate for broadcasters, as they are more active in the media platform, could be taken as much smaller than μ (we can call it μ_2), while the rate at which inerts leave the population could be a bit more than μ, which could be taken as μ_3. A model considering that will have a variable population instead of a fixed one. We tested our model results with a typical set of parameters maintaining the above logical relation, with $\mu = 0.05$, $\mu_1 = 0.03$, $\mu_2 = 0.005$ and $\mu_3 = 0.07$. With the other parameters unchanged, the equilibrium analysis shows no qualitatively different results; bistable (for $\alpha = 1$) as well as monostable (for $\alpha = 0.1$) dynamics were observed for higher and lower values of α respectively, but the mathematical handling becomes complicated.

The model presented in this paper is the first to include a relapse rate while analyzing epidemic spread and sustainability of viral marketing messages. The remarkable effect that this relapse rate has on the sustainability of the campaign has deeper marketer-level implications. Through last couple of years advertisers have slowly understood the importance of capturing the attention of lost customers. We are already familiar with Facebook retargeting for products, where by adding a code snippet (often called a *pixel*), the online websites retarget attention of the customers, whom they lost from their website due to unknown reasons [45]. Firms are becoming quite inclined to get the services of companies like Adroll, Retargeter, Perfect Audience, etc. or going directly to the exchanges like Google, Facebook, Twitter [46] for running their own retargeting campaigns to re-engage anonymous users. But recent studies show that continuous retargeting leads to a definite privacy concern and skepticism among customers, which results into a lower purchase intention [47]. Our findings in this paper points out the relapse can cause a major effect, especially if a social-circle-level remarketing technique can be devised where factors like authenticity and security will be highlighted. The campaigns should adopt clear privacy policies about protection of consumer data, as well as consider adding a social context to encourage spontaneous reminders among the population. As friends and peers have a substantial influence, close proximity and often share similar interest, it is both more plausible and effective, if they assure the lost customers about the genuineness and usefulness of a campaign. To address this peer effect, in a future project, it will be valuable to know the existence and nature of steady states in adaptive as well as weighted networks. Different rules of adaptation can be employed to understand the behavior of diffusion in more realistic scenarios.

Acknowledgment

The authors would like to thank Prof. Yatindra Nath Singh for his valuable comments, as well as for providing us with some essential computational facilities.

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